

# **The Numerical Solution of Underwater Acoustic Propagation Problems Using Finite Difference and Finite Element Methods**

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July 9, 1984

This work was done while the author was visiting the Large Aperture Acoustics Branch of the Acoustics Division at NRL while on leave from the Applied Mathematics Department at Brookhaven National Laboratory.



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<p>This report discusses various aspects of the numerical solution of underwater acoustic wave propagation problems. In the first part of the report, a model propagation problem based on the two-dimensional Helmholtz equation with a variable sound speed is considered. A finite element computer code for solving such problems was implemented at NRL on the VAX 11/780. A distinctive feature of the code is the implementation of a recently developed iterative method for solving the resulting large, sparse, indefinite, non-self-adjoint system of equations. This allowed for the efficient solution of over 35,000 complex equations on a relatively small computer. Some of the results obtained after applying this code to the model problem are described. Furthermore, additional modifications that can be made to the code to improve its efficiency and extend its applicability to more general propagation models are</p> <p style="text-align: right;">(Continued)</p>				
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discussed. In the second part of this report, the general situation of the coupled acoustic/elastic wave equation in two and three dimensions is considered. For example, this may correspond to an ocean environment in which there is ice on the surface as well as an irregularly shaped bottom structure. Finite difference and finite element methods for solving both the time harmonic and time dependent models are discussed. Various issues are considered that are important in determining the size of the problem that can be adequately treated. This includes the computer power as well as the mathematical and modeling techniques available.

## PREFACE

This report outlines recent efforts to use finite element techniques for solving the wave equation of underwater acoustic propagation in large ocean regimes. This effort is a part of a computationally intensive probabilistic acoustics program whose major current goal is to develop models for the propagation of the moments of the acoustic field in regions where the boundaries are random surfaces. With the rapidly increasing computer power of large main-frame computers, indeed even large minicomputers augmented with powerful array processors, the possibility of using an exact technique as the finite element method (FEM) for solving the wave equation, even in such complicated regimes, needs to be addressed. In the near term this method is expected to be especially applicable for benchmark or exact-solution calculations subsequently used to evaluate large complex computer codes, since the length of time that such calculations take is not, in general, the governing parameter. Future advances in computer design can make the FEM also attractive for solving problems in realistic scenarios and for use in a production mode. For relatively small source-to-receiver ranges at low frequency, the FEM can already be used as a predictor of transmission loss in a complicated environment.

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# THE NUMERICAL SOLUTION OF UNDERWATER ACOUSTIC PROPAGATION PROBLEMS USING FINITE DIFFERENCE AND FINITE ELEMENT METHODS

## INTRODUCTION

In this report, we discuss various aspects of the numerical solution of underwater acoustic-wave propagation problems. The propagation of acoustic energy in the ocean involves the interaction between acoustic-wave propagation in fluids and stress-wave propagation in solids. Thus, a general mathematical model involves the coupling of the acoustic-wave equation with the elastic-wave equation and the specification of suitable interface and boundary conditions. Only simple wave-propagation problems can be solved analytically. Hence, the approximate solution of time-harmonic and time-dependent models in two and three dimensions is important to treat effectively acoustic propagation in a general ocean environment. Note that we are considering linear, forward, deterministic propagation problems here. However, many of the methods may also be applicable to nonlinear, inverse, and stochastic problems.

Various computational approaches have been developed and applied to simplified propagation models. These include parabolic-equation and normal-mode models, asymptotic methods, and others; see, for example, Ref. 1 for a survey of various models and numerical techniques. Although each of these techniques can be quite effective under suitable assumptions, there are many important problems for which it is necessary to treat the complete wave-propagation model described above in the low to intermediate frequency range. Such models can include lateral inhomogeneities, multiple irregular interfaces and boundaries, full angle propagation, and backscattering. This occurs, for example, when the ocean bottom must be taken into account, such as in shallow-water propagation and in deep water at very low frequencies. The interaction of acoustic and seismic waves with a complicated ocean bottom is an important and difficult problem. Another important example of a complicated propagation problem occurs when a layer of ice is present on the ocean surface.

Finite difference and finite element methods have proved to be very effective techniques for solving approximately boundary and initial-value problems of the type described above. However, to properly resolve the waves it is necessary to decrease the spatial mesh sizes as the frequency increases. This can result in problems with very large numbers of degrees of freedom as the frequency and/or spatial dimensions increase. The size of the problems that can be effectively treated numerically depends on various factors, such as the computer power, as well as the mathematical and modeling techniques available. We shall discuss these and other issues in more detail later and consider various possibilities for treating large, complicated ocean-propagation problems more efficiently.

We close this section by outlining the remainder of this report. In the second section, we consider a model problem involving the two-dimensional Helmholtz equation with a variable sound speed; this was treated using a finite-element code implemented at NRL on the VAX 11/780 and modified to treat underwater acoustic-propagation problems. We briefly describe some features of the finite-element algorithm and the treatment of radiation boundary conditions. A distinctive feature of the code is the implementation of a recently developed iterative method [2] for solving the resulting large, sparse, indefinite, non-self-adjoint system of equations. This allowed for the efficient solution of over 35,000 complex equations on a relatively small computer, since large matrices did not have to be stored

or inverted. We describe some of the numerical results obtained after applying this code to the model problem. Furthermore, we discuss additional modifications that can be made to the code to improve its efficiency and extend its applicability to more general propagation models.

In the third section, we consider the general situation of the coupled acoustic/elastic wave equation in two and three dimensions. We discuss finite-difference and finite-element methods for solving both the time-harmonic and time-dependent model. The numerical techniques for solving the time-harmonic and time-dependent problems are very different, as are the numerical difficulties that are encountered. Various issues are considered that are important in determining the size of the problem that can be adequately treated. Finally, in the fourth section we summarize our findings.

## A MODEL PROBLEM

In this section we describe a model underwater acoustic propagation problem based on the two-dimensional Helmholtz equation with a variable sound speed. We also describe results obtained after implementing a finite-element code and modifying it to treat this propagation model. Finally, we indicate several ways of improving the capabilities of this code.

### The Model and the Numerical Algorithm

The Helmholtz equation for a cylindrically symmetric geometry and a harmonic source is given by

$$\Delta u(r,z) + K_0^2 n^2(r,z) u(r,z) = 0,$$

where  $z$  is the depth measured downward from the ocean surface,  $r$  is the range,  $u(r,z)$  is the acoustic pressure,

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2}$$

is the Laplacian in cylindrical coordinates, the reference wave number is  $K_0 = 2\pi f/c_0$ ,  $f$  is the source frequency,  $c_0$  is a reference sound speed, the refractive index  $n(r,z) = c_0/c(r,z)$ , and  $c(r,z)$  is the sound speed. For simplicity, we consider a region of the ocean with a flat surface and a flat bottom, so that the region  $D$  is a rectangle (see Fig. 1). We assume an ocean depth of 5000 m, an initial range given by  $r = R1$ , and a final range given by  $r = R2$ .

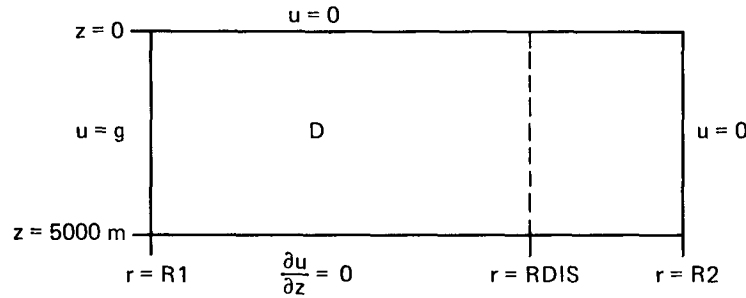


Fig. 1 — Region for the model problem

Our boundary conditions are given by

$$\text{Top: } u(r, 0) = 0, \tag{1a}$$

$$\text{Bottom: } \frac{\partial u}{\partial z}(r, 5000) = 0, \tag{1b}$$

and

$$\text{Left side: } u(R1, z) = g(z), \quad (1c)$$

where  $g(z)$  is a specified initial pressure field. The boundary condition at  $r = R2$  is chosen so as to model the outgoing radiation of energy. Since there is no backscattering, we may model this outgoing radiation condition by choosing a suitable artificial dissipation,  $id(r)$ , in the region  $RDIS \leq r \leq R2$ , where  $d(r)$  is a real-valued nonnegative function to be specified later. The right-hand boundary condition is now defined by

$$\text{Right side: } u(R2, z) = 0. \quad (1d)$$

We shall see later that there are various alternative methods for modeling the outgoing radiation condition.

Our goal is to solve approximately the following elliptic partial differential equation in the region D, combined with boundary conditions (1a) through (1d):

$$\Delta u(r, z) + K_0^2 n^2(r, z) [1 + id(r)] u(r, z) = 0. \quad (2)$$

We have run our computer code with the following choices of parameters and functions:

$$g(z) = \frac{\sqrt{K_0}}{2} \left\{ \exp \left[ -K_0^2 (z-z_0)^2/4 \right] - \exp \left[ -K_0^2 (z+z_0)^2/4 \right] \right\},$$

$c_0 = 1500$  m/s, source depth  $z_0 = 2500$  m,

$$\frac{1}{c^2(z)} = \frac{1}{c_1^2} - a^2(z-z_1)^2, \quad a = 10^{-7} \text{ m}^2/\text{s},$$

depth of sound speed minimum  $z_1 = 2500$  m,  $c_1 = 1500$  m/s,  $R1 = 1$  m, and  $R2$  has been chosen thus far between 3000 and 25,000 m. The frequency  $f$  has been chosen thus far between 3 and 10 Hz. This particular index of refraction causes a "focus" at about 20 km. Note that while the initial Gaussian pressure field  $g(z)$  does not exactly satisfy boundary conditions (1a) and (1b), these boundary conditions are sufficiently closely satisfied for our choice of parameters that this causes no computational difficulties. This model was previously run using a parabolic equation method implemented at NRL [3], although the bottom-boundary condition in Ref. 3 differs from Eq. (1b). We have found empirically that a convenient functional form for the dissipation term,  $id(r)$ , is given by

$$d(r) = \begin{cases} e^{\beta(r - RDIS)} - 1 & \text{for } RDIS \leq r \leq R2, \\ 0, & \text{for } r \leq RDIS, \end{cases}$$

with  $\beta$  and  $RDIS$  suitably chosen, although other simple functions can work about as well. The purpose of this dissipation term is to attenuate the wave while at the same time minimizing the reflection due to the dissipation. Note: There is no difficulty in treating range- and depth-dependent sound speeds. Also,  $R1$  can be chosen quite large. The input data,  $g$ , may be specified as the result of a (long-range) parabolic equation run or some other numerical or asymptotic method. Finally, the model problem may be readily generalized to include complicated geometries, boundary conditions, and interfaces. This latter point is discussed in more detail later.

The numerical algorithm we have employed to solve approximately the boundary-value problem given by Eqs. (1) and (2) is based on the *finite element method*. We shall not go into a technical discussion of the finite-element method, since it is described extensively in the literature (see, e.g., Ref. 4). We merely point out that the finite element method is based on replacing the given boundary-value problem by an equivalent variational problem and then approximating the variational problem by use of a convenient finite-dimensional space of functions. Typically, this space of functions consists of sufficiently smooth piecewise polynomials defined with respect to a partitioning of the computational



domain into simple subsets called elements. This reduces the variational problem to that of solving a finite number of linear equations. As the diameter of the largest element decreases, the approximate solution converges to the exact solution, but the number of equations to be solved increases. The computer code we have implemented to solve the problem is based on continuous piecewise linear functions defined on right triangles. The code can use either uniform or variable mesh sizes. Let us observe that finite difference methods can be used just as effectively to solve the problem approximately. Finite element methods, however, are more effective for treating complicated boundaries, interfaces, and boundary conditions.

An important, distinctive feature of our numerical algorithm and computer code is the incorporation of a recently developed *iterative method* [2] for solving the resulting sparse system of linear equations. The solution of this system of equations is the most expensive part of the computation. It is well-known that iterative methods are in general considerably more efficient than direct methods (i.e., those based on Gaussian elimination) for large problems, with respect to both storage and computational speed. However, iterative methods have typically been developed and analyzed for positive definite, symmetric problems (see, e.g., Ref. 5). Neither of these properties holds for the problems currently under investigation. An iterative method based on the preconditioned conjugate gradient method was described in Ref. 2 for a class of problems including the time-harmonic problems discussed in this report. The preconditioner is based on one sweep of symmetric successive overrelaxation (SSOR), although other preconditioners are being investigated. Its implementation has resulted in a dramatic increase in storage capabilities and a dramatic decrease in computer time with respect to direct solvers, since large, sparse matrices do not have to be stored or inverted.

## Numerical Results

We next briefly describe results obtained after applying the finite element code to the boundary-value problem given by Eqs. (1) and (2). This work concentrated on the following:

- Implementing a finite element code developed elsewhere on the VAX 11/780 in the Large Aperture Acoustics Branch at the Naval Research Laboratory and modifying the code to treat propagation problems such as the aforementioned one;
- Improving the code so as to increase its speed and, particularly, its storage capabilities; and
- Testing important quantities, such as the range of artificial dissipation and the number of grid points/wavelength needed for a prescribed accuracy, as well as the CPU time for various choices of parameters.

It is important to emphasize that we are solving large problems on a relatively small computer. We are now able to store and solve over 35,000 complex equations on the VAX 11/780. (When the code was originally implemented, we were limited to about 12,000 equations.) Furthermore, the subroutines used for the iterative method have been improved so as to make them more efficient for use on a vector computer or an array processor. There is still a great deal that can be done to increase the efficiency of the code. This is outlined later.

To determine the length of artificial dissipation and the number of grid points/wavelength needed, we used measures based on the average volume intensity and average line intensity of the computed solution. We can plot the solution as a surface, and we can plot the transmission loss vs range. We have determined empirically, using these measures and plots, that when the length of the dissipation layer is about two to four wavelengths, there is no significant deterioration in the solution. As for the resolution of the waves, we observed that a minimum of 8 grid points/wavelength is necessary to obtain a meaningful solution when a uniform grid size is used.

To get an idea of the CPU time for typical runs, we consider the following sample test runs at 3, 5, and 10 Hz for  $R2 = 2001$  m, using approximately 8 and 12 grid points/wavelength (see Table 1). Note that as the frequency,  $f$ , is multiplied by a factor  $C$ , the number of equations is multiplied by approximately  $C^2$  (to resolve the waves in two dimensions). Furthermore, the number of iterations generally increases only slightly. This gives a rough idea of how the CPU time increases with frequency for fixed range. We also observe that as the mesh size decreases from a coarse mesh (8 points/wavelength) to a finer mesh (12 points/wavelength), the number of iterations increases very slightly or can even decrease even though the number of equations increases. This occurs because the system of equations is better conditioned as the discrete model better approximates the continuous model. Further study is needed to assess the accuracy of the discrete model with respect to changes in all the parameters.

Table 1

Case	$f$ (Hz)	Points/ Wavelength	Number of Equations	Number of Iterations	CPU Time (min)
(a)	3	8	2,673	169	8
(b)	5	8	7,809	287	39
(c)	10	8	28,944	287	150
(d)	3	12	5,929	196	21
(e)	5	12	17,425	216	84

Next, suppose that the frequency and grid points/wavelength are fixed but the range increases by a factor  $C$ . Hence, the time for each iteration increases by a factor  $C$ . Furthermore, the number of iterations will increase typically by a factor  $C'$ , where  $1 < C' < C$ . For example, consider case (a) in Table 1 (3 Hz over 2000 m). We ran it over 25,000 m, so that  $N = 32,481$ . The CPU time was 414 min. (The number of iterations was multiplied by a factor of 4.) In terms of storage on the VAX, the current version of the code can treat 5 Hz over 10,000 m or 10 Hz over 2500 m using a uniform mesh.

We next describe a *factorization-mesh grading* procedure developed to treat longer ranges without increasing the storage capabilities. This method consists first of factoring out  $e^{ik_0 r}$  from the solution. This results in a smoother solution as  $r$  increases and thus allows for longer range step sizes as we proceed away from the origin. This is consistent with the approach taken when the parabolic equation method is employed, where it is typically observed that range steps several wavelengths long suffice for accurate far-field solutions. This approach is also analogous to a method developed and analyzed in Ref. 6 in connection with the Helmholtz equation exterior to a bounded obstacle. In the current code, we implemented this factorization-mesh grading procedure using larger range steps with increasing  $r$ . Due to lack of time, we were unable to study its effectiveness comprehensively. However, preliminary results indicated that we could substantially reduce the number of grid points without losing accuracy. For example, we ran a problem with 3 Hz over 30,000 m using this procedure and were able to reduce the number of range points by nearly a factor of three compared to a uniform grid with 8 points/wavelength.

### Suggested Improvements and Extensions

We conclude this section by outlining several modifications that can be made to increase the capabilities of the present code and shed more light on its efficiency and generality.

(1) There is no difficulty in changing the initial pressure field,  $g(z)$ , and sound-speed profile,  $c(r,z)$ , in the computer code. Hence the code could be readily employed to solve a propagation problem for which the exact solution is known. For example,  $g(z)$  could be generated by use of a normal-mode solution. This could be useful in obtaining more detailed information regarding the accuracy of

the solution as important parameters are varied. Furthermore, the factorization-mesh grading procedure could be more accurately assessed since  $g(z)$  could be made to correspond to either a near- or far-field solution. In the latter case, for example with one dominant propagation mode, we could see how large the range steps can be made for the far-field solution. This would also enable us to study a model problem at higher frequencies over longer ranges.

(2) The most interesting modification in the code that could be made in a relatively straightforward way would be to put in an ocean-bottom structure such as a seamount. The seamount can be modeled by a jump in the density and sound speed across an interface, assuming the absence of shear waves. Observe that layers in the bottom structure could be treated in an analogous fashion. The finite element method is well suited to treat interface conditions of this kind.

To be precise, consider the interface  $\Gamma$  separating the ocean (region D1) and seamount (region D2), illustrated in Fig. 2. We assume that the density  $\rho_i$  is a constant in the region  $D_i$  and denote the pressure  $u$ , and sound speed  $c(r,z)$ , restricted to  $D_i$  by  $u_i(r,z)$  and  $c_i(r,z)$ ,  $i = 1, 2$ . The following two interface conditions must be satisfied at each point  $(r,z)$  on  $\Gamma$ :

$$u_1(r,z) = u_2(r,z) \quad (3a)$$

and

$$\rho_1^{-1} \frac{\partial u_1(r,z)}{\partial n} = \rho_2^{-1} \frac{\partial u_2(r,z)}{\partial n}, \quad (3b)$$

where  $\partial u / \partial n$  denotes the normal derivative in the direction  $n$  shown in Fig. 2. A complete mathematical model for this situation may now be given by the boundary-value problem of Eqs. (1) and (2) combined with interface conditions (3a) and (3b). Note: If  $\rho_1/\rho_2 = 0$ , this problem is reduced to a boundary-value problem on the computationally smaller domain, D1, with boundary condition on  $\Gamma$  given by  $\partial u_1 / \partial n = 0$ . Furthermore, if  $\rho_2 \gg \rho_1$  it is readily seen that very little energy is transmitted into region D2 (see, e.g., Ref. 7). Hence, we expect in this case that relatively large mesh sizes might suffice in region D2.

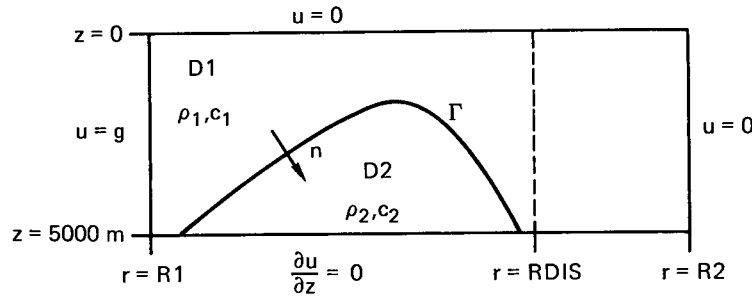


Fig. 2 — Region containing an interface

It would be quite informative to assess some of the modeling and computational aspects of the present algorithm with respect to this boundary-value problem. For example, it would be interesting to see how well the artificial dissipation models the outgoing-radiation condition. For this to be effective, there must be no backscattering occurring in the dissipation layer. It would also be interesting to evaluate the effectiveness of the iterative method and the use of variable range step sizes (after factoring out  $e^{iK_0 r}$ ) for this model. The boundary data,  $g$ , in  $r = R1$  might be prescribed by use of a normal-mode solution as discussed in paragraph (1) above or by propagation with the parabolic-equation method up to that range. Finally, note that this model is a special case of the time-harmonic fluid-solid interaction model to be discussed in the next section.

(3) There are various changes that could be made in the code to increase the storage capabilities and decrease computation time. For example, higher frequencies and longer ranges could be treated by subdividing the range into several subregions and solving the problem successively over each subregion, with the computed solution over the  $i$ th subregion used as boundary data for the  $(i + 1)$ st subregion. The applicability of this method may depend on sufficiently accurate solutions being obtained.

Another potential improvement in the code involves the use of a "lumping" procedure to reduce the number of nonzero matrix elements per row from seven to five. In this case, all elements except the diagonal could be made real. This would eliminate several large vectors of storage and probably allow for the solution of close to 50,000 equations on the VAX. Furthermore, the computer time would probably be reduced because of the reduction in size of several large arrays. The effect on the convergence rate of the lumping is not clear and should be investigated.

(4) Some simple modifications in a few subroutines would put the code in a form suitable for an array processor. This could potentially speed up the calculations considerably, provided a large enough array processor is available. Also, since the code is, for the most part, vectorizable one can take advantage of the vector capabilities of vector computers such as the CRAY-1 and CYBER 205.

Of the changes suggested here, we feel that (1) and (2) are the most important ones to be made first. In addition, there are a number of mathematical techniques that could lead to significant improvements in the computer code. These are of a longer term nature and are discussed in the next section.

## GENERAL PROPAGATION MODELS

In this section we investigate various computational aspects of two- and three-dimensional time-harmonic and transient underwater acoustic propagation models. We consider a general ocean environment in which the acoustic-wave equation in the liquid is coupled with the elastic-wave equation in the solid. Hence, both dilatational and shear waves are permitted in the solid. For generality, let us assume a solid structure on both the ocean surface and the bottom. For example, this may correspond to a situation in which there is ice on the surface as well as an irregularly shaped bottom structure.

There are several mathematical formulations of the partial differential equations modeling the fluid-solid interaction. One convenient model, discussed in Ref. 8, uses a displacement formulation of the elastic-wave equation in both the fluid and solid regions. In the fluid region, the shear modulus is set equal to zero. Hence, in this case the model consists of a coupled system of partial differential equations for the unknown vector displacement. Furthermore, certain interface conditions must be imposed at the fluid-solid boundaries. These correspond to the condition that the normal components of the stress and displacement and the tangential component of the stress must be continuous across the interfaces [7]. There are alternative formulations that can also be employed to model this situation. For example, the fluid can be modeled by use of a pressure formulation or a mixed formulation in which both the pressure and displacement vector are taken as unknowns. For a more detailed discussion of various alternative formulations, as well as a comprehensive list of references, see Ref. 9.

In the time-dependent formulation, the model is a hyperbolic initial boundary value problem and certain initial conditions must be specified. In the time-harmonic formulation, the model is an elliptic boundary value problem. In either case, there are various ways in which we can prescribe boundary conditions on the four sides. (See Fig. 2 for an example of the type of region we are discussing.) For the surface of the fluid, a zero pressure condition is generally imposed. The bottom and surface boundary conditions corresponding to the solid structure can be modeled in different ways, depending on the degree of detail desired. See, e.g., Ref. 9 for different ways of modeling the bottom.

The two vertical boundary conditions can also be modeled in different ways. For the left boundary condition we might assume, as in the previous section, that the field is specified, perhaps as the result of a long-range run with a simplified propagation model used. Alternatively, the loading might be such that a plane of symmetry exists. In this case the particle motions are zero normal to the plane of symmetry on this boundary. The right vertical boundary requires the imposition of an outgoing-radiation (nonreflecting) boundary condition. This is an active area of research and can have an important impact on the computations. We shall consider alternative formulations of this radiation condition later.

The resulting computational model can be made discrete by the use of either finite difference or finite element techniques. This typically results in a large, sparse system of linear equations. The problem size depends mainly on the frequency and the size of the computational domain. If we assume for simplicity a uniform grid size, it is clear that if we multiply the length of the domain by  $m$ , then the number of equations is multiplied by  $m$ . However, multiplying the frequency by  $m$  results in multiplying the number of equations by at least  $m^2$  and  $m^3$  in two and three dimensions, respectively, since the waves must be resolved in each spatial direction. The frequencies and dimensions of interest depend, of course, on the particular physical problem.

We thus see that the problem size can be quite large, particularly as the frequency increases. (Asymptotic methods, such as ray tracing, can be effective for high frequencies. However, it is not clear in general for which frequencies these methods yield reliable results.) In the remainder of this section, we discuss various issues relating to the size of the propagation models that might reasonably be solved numerically. We first consider the computer power that is currently available and that is expected to become available in the next few years. We then discuss some mathematical techniques that can have an important bearing on the solution of the computational models. The mathematical methods for treating time-harmonic and transient models are very different and will be discussed separately.

### Computational Power

Because of the increasing need for large-scale, high-speed computers in many different areas of science and engineering, there is a great deal of work proceeding with the aim of greatly increasing computing power. We first briefly discuss the increased computer power obtainable using vector machines such as the CRAY-1 or CYBER 205. We observe in this connection that the finite-element algorithm we described previously is, for the most part, vectorizable. The main exception is the SSOR preconditioner used in the iterative method. We discuss later alternative preconditioners whose implementation can make the iterative method completely vectorizable. We will attempt to compare the computational power of the CRAY-1, CRAY X-MP, and CRAY-2 with that of the VAX 11/780 used for the computations discussed in the previous section. These comparisons are based on the best knowledge and conjectures available to the author at this time.

The CRAY-1, currently in use, has a memory of between  $2 \times 10^6$  and  $4 \times 10^6$  64-bit words. The memory available to us on the VAX was about  $1 \times 10^6$  32-bit words. Note that the use of 64 bits on the CRAY is roughly equivalent to the use of double precision on the VAX. The CRAY-1 is approximately 40 times as fast as the VAX. The CRAY X-MP is scheduled to be operating sometime in 1984 and is to be about twice as fast as the CRAY-1. Finally, the CRAY-2 is to be a substantial improvement over the CRAY-1 in both memory and speed. Specific details at this point are not known. However, available information indicates an improvement in speed of up to a factor of ten compared to the CRAY-1 and a memory of  $256 \times 10^6$  64-bit words. We understand that it is scheduled to be available before the end of 1985. We also understand that CDC is planning a successor to the CYBER 205, by 1986, to consist of several processors, each processor three to five times as fast as the 205 processor.

There are intense efforts under way to increase computing power by two or three orders of magnitude. These research efforts are based on the concept of parallel computing using several different processors. There is an abundance of concepts for parallel computing systems as well as special purpose computers (such as finite element machines). We shall not go into these issues but instead refer to Ref. 10 and the references cited there for technical details. We merely remark that this research is of a long-term nature and it is not clear when parallel computing systems will be available to deliver this kind of computing power for realistic problems.

Let us next try to assess the feasibility of solving large problems efficiently on both the CRAY-1 and the CRAY-2. The following discussion is based on estimates obtained from a simple scaling of problems which have already been solved. It is not possible to predict how valid this scaling is.

For the purpose of this discussion, we consider the Helmholtz model described previously over a range of 40 km at a frequency of 40 Hz. Furthermore, we assume a uniform grid size of eight points/wavelength although, as we have seen, we could considerably reduce the number of equations using nonuniform grid sizes. We see from Table 1 Case (c) that for a frequency of 10 Hz and a range of 2 km the CPU time on the VAX was 2.5 h. Let us now simply scale up this problem to 40 Hz over 40 km multiplying by  $4^2$  for the frequency increase and 20 for the range increase, using the reasoning outlined above. We would then obtain for the large problem (consisting of nearly  $10^7$  complex degrees of freedom), a CPU time of 800 h on the VAX, 20 h on the CRAY-1 and 2 h on the CRAY-2 (assuming for the sake of definiteness that the CRAY-2 will be a factor of 10 faster than the CRAY-1). This rough scaling neglects any increase in iterations as the size of the region increases.

There are a number of factors that cannot be predicted in advance. For one thing, it is not really possible to predict how a given code will behave on a different computer. Furthermore, the problem is too large to fit in the central memory of the CRAY-1. This necessitates an input/output process that can cause a considerable increase in time. This additional time can be minimized by the use of a numerical algorithm well designed to deal with this difficulty. This problem would be considerably less severe for the CRAY-2, however, because of its predicted large central memory, if we assume that a large part of its memory is available for this problem. Indications are that memory is coming down in price, so it is to be expected that a wide range of computers will have more memory available to them.

Finally, it is anticipated that the suggested improvements outlined in the previous section and, more important, those improvements described in the following would lead to substantial improvements in the algorithm and computer code. It is not possible, however, to predict the exact degree to which such improvements would reduce the storage requirements and CPU time. For these and other reasons, the only way adequately to assess the parameter ranges that can be efficiently modeled is by means of continuing numerical studies employing the best modeling and mathematical techniques available.

### Time-Harmonic Models

We now discuss mathematical models corresponding to a time-harmonic source. Such models commonly occur in underwater acoustic propagation problems [1]. Since the harmonic time dependence is factored out, we are left with an elliptic boundary-value problem. This problem consists of a system of coupled second-order elliptic partial differential equations, each analogous to the Helmholtz equation considered previously. We proceed to discuss briefly three areas of mathematical research that can have an important bearing on the numerical solution of these models. This is not intended to be a comprehensive survey but merely to indicate some of the research directions that can yield fruitful results.

### *Radiation Boundary Conditions*

To solve the problem approximately, we must introduce an artificial vertical boundary on the right side of the computational domain (see Fig. 2) as well as an appropriate radiation (absorbing) boundary condition to simulate the outgoing propagation of energy. This artificial boundary can intersect the fluid domain completely, the solid domain completely, or both the fluid and the solid domains. Various methods have been formulated for modeling radiation boundary conditions, and much research is continuing in this area. The goal is to formulate a boundary condition that minimizes spurious reflections that can contaminate the solution without increasing the size of the computational domain unnecessarily. At the same time, it is important for this formulation to be suitable for efficient implementation and solution. The choice of the most efficient formulation is to a great extent problem dependent. For our model problem in the previous section, we discussed the use of artificial dissipation to attenuate the wave. This method is only applicable when no backscattering occurs. Empirical evidence indicates that in such cases the method can be effective with only a few wavelengths of artificial dissipation.

We first assume that the artificial boundary intersects only the fluid domain, so that we consider the acoustic-wave equation. In many cases, the radiation condition may be expressed in terms of a modal expansion for the outgoing solution. For example, for the case of a rotationally symmetric medium the following radiation condition was employed in combination with the finite element method [11]:

$$\frac{\partial u}{\partial n} = T(u). \quad (4)$$

Here, the expression for  $T(u)$  involves a series expansion in Hankel functions to represent outgoing waves at the artificial boundary and  $\partial u / \partial n$  is the normal derivative at this boundary. This formulation is valid whenever the problem is separable. This occurs, for example when the sound speed is range independent and both the upper and lower boundaries are horizontal. It is only necessary to consider a finite number of terms in this expansion corresponding to the propagating modes. An analogous condition was analyzed in Ref. 12 in connection with a variety of geometries, and it was shown that the finite element method converges optimally in spite of the complicated nature of this boundary condition.

A hierarchy of boundary conditions based on this modal expansion was employed in Ref. 2. Each successive boundary condition in this hierarchy was constructed to be exact for successively more propagation modes. Unlike condition (4), these boundary conditions are local (i.e., the normal derivative of  $u$  at a point is given in terms of  $u$  and its tangential derivatives at that point). Hence the resulting system of equations is sparse, which leads to certain computational advantages. When there are only one or two propagating modes, these boundary conditions are more conveniently implemented than Eq. (4). However, practical difficulties occur in the implementation of higher order boundary conditions due to the presence of higher order tangential derivatives. The global condition (4) is applicable even when there are many propagating modes.

Another hierarchy of local boundary conditions was developed and analyzed in Ref. 13 for the time-dependent wave equation using pseudodifferential operators. These conditions are also applicable to the Helmholtz equation [14]. The first boundary condition in this hierarchy is closely related to the viscous boundary condition, described in Eq. (5). It is exact if the solution is a plane wave normally incident to the artificial boundary. The higher order boundary conditions in Ref. 13 are designed to give better approximations as the deviation from normal incidence increases. As before, however, there are computational difficulties in implementing the high-order boundary conditions, since they involve high-order derivatives.

Next, suppose that the artificial boundary intersects the solid domain. A convenient method for handling the absorbing boundary condition in conjunction with the displacement formulation of the elliptic wave equation was described in Ref. 15. A plane wave boundary condition of the form

$$\sigma_n = \rho c_d v_n, \quad \sigma_t = \rho c_s v_t \quad (5)$$

is applied at the artificial boundary. Here,  $\sigma_n$  and  $\sigma_t$  are the normal and tangential interface stresses,  $\rho$  is the solid-medium mass density,  $c_d$  and  $c_s$  are the dilatational and shear wave speeds, and  $v_n$  and  $v_t$  are the corresponding normal and tangential velocities on the solid boundary surface. Conditions (5) are exact if the radiating energy at the surface consists of plane waves that are normally incident to the surface. It was pointed out in Ref. 15 that these boundary conditions can still be accurate even when this condition does not hold. The artificial boundary acts as if viscous dampers have been applied normally and tangentially to the solid-domain boundary points. When the dominant waves traveling through the media were Rayleigh waves rather than plane waves, an analogous boundary condition was applied in Refs. 15 and 16. The boundary conditions in Eq. (5) were also applied to treat the case in which the artificial boundary intersects both the fluid and solid media [9]. Empirical evidence indicates that these boundary conditions can give satisfactory results if they are located far enough from the source force functions [17,18].

Finally, we note that there are various alternative methods for treating the unbounded region, such as the use of *infinite elements* [19]. In this case, the usual piecewise polynomial basis functions are replaced by special functions in the outermost layer of elements. These infinite elements are chosen so as to simulate the behavior of the solution near infinity. Another approach for treating unbounded domains consists of coupling the finite element method with an integral operator on the outer boundary [20]. The Green's function for the integral operator simulates the behavior of the solution near infinity. For additional methods for treating radiation boundary conditions, see Refs. 9 and 14.

#### *Adaptive Discretization Methods*

To solve the resulting elliptic boundary-value problem approximately, we must discretize this problem. Generally, this may be efficiently done by a variety of finite difference or finite element methods. Finite element methods are better suited to treating the complicated boundaries and boundary conditions that often arise in the propagation problems considered here. Because of their great flexibility in modeling complicated problems, we recommend the use of finite element methods. It is not clear, at the present time, whether high-order or low-order methods would be more efficient for these problems. This question needs to be investigated further.

Adaptive computational methods have recently proved to be a powerful tool in the numerical solution of partial differential equations, and considerable research is continuing along these lines. These algorithms have built into them convenient methods for obtaining measures of accuracy and adapting the discretization automatically to the evolving solution, so as to obtain a desired level of accuracy with a minimum of arithmetic operations. These algorithms automatically provide for smaller mesh sizes in a region where the solution is not smooth (e.g., where there is a large velocity or density gradient) and larger mesh sizes when the solution is smooth. Recently developed adaptive discretization methods have resulted in significant improvements in the numerical solution of elliptic boundary-value problems in elasticity and other areas (see Refs. 21 and 22, and the references cited there.)

We feel that the use of adaptive methods can also lead to dramatic improvement in the numerical solution of underwater acoustic propagation problems. As we have demonstrated (in a previous section, in Ref. 6, and in Ref. 23), the use of nonuniform grid sizes in connection with the Helmholtz equation can lead to improved accuracy with considerably fewer grid points. Adaptive methods provide an efficient means of changing the mesh sizes (and other important parameters) systematically in a nearly optimal manner. Furthermore, the adaptive method provides a means for assessing the accuracy of the computed solution. Hence, another potential advantage is that the error due to various simplified propagation models may be conveniently determined.



### *Iterative Methods*

The finite element or finite difference discretization of our model results in a sparse, indefinite, non-self-adjoint system of linear equations. As we have mentioned previously, iterative solution methods are far superior to direct methods for very large problems. However, standard iterative methods are not applicable to systems that are indefinite and non-self-adjoint. The iterative method discussed previously in connection with the Helmholtz equation is also applicable to the more-general boundary-value problems considered here. It is based on the conjugate gradient method applied to the normal equations with an appropriate preconditioning (see Ref. 2). The performance of the iterative method has a significant bearing on the cost of the computation. Hence, the study of preconditioned conjugate gradient methods and other iterative methods in connection with elliptic boundary-value problems is receiving a great deal of attention (see, e.g., Refs. 24 and 25). We shall briefly consider ways of potentially improving the iterative method described previously.

The choice of preconditioner in the preconditioned conjugate gradient method is crucial in determining the number of iterations required for a desired accuracy and, hence, the amount of computation. Another important factor is the method in which the preconditioner is implemented in the conjugate gradient method. Thus far we have only implemented a preconditioner based on one sweep of SSOR in connection with the problem of Eqs. (1) and (2). As demonstrated in Refs. 2 and 26, the number of iterations using this preconditioner grows as  $O(h^{-1})$  as the mesh size  $h \rightarrow 0$ . There are several alternative preconditioners that have proved to be quite effective in connection with the conjugate gradient method [24]. Some of these are currently being investigated in connection with the Helmholtz equation [26], and in theory appear to exhibit a faster convergence rate than SSOR for small mesh sizes. For example, it can be seen theoretically that a preconditioner based on a fast solver (such as a multigrid method) has a convergence rate that is independent of  $h$ . Hence the number of iterations is independent of  $h$  and such a preconditioner would appear to be superior for sufficiently small mesh sizes.

It remains to be seen whether preconditioners that are more effective as  $h \rightarrow 0$  will be more effective in realistic underwater acoustic problems. There are many other factors that play a role in the choice of preconditioners, such as their behavior with respect to the wave number and boundary conditions. Finally, we recall that, to utilize fully the capabilities of vector computers such as the CRAY-1 or CYBER 205, the algorithm should be vectorizable. This is not the case of the preconditioners we have implemented thus far. For examples of preconditioned conjugate gradient methods that have been vectorized, see Refs. 27 and 28. The choice of an optimal preconditioner for a given class of problems and a given computer can be complicated, but it can be important in reducing the cost of the computation.

### **Time-Dependent Models**

Time-dependent models are most appropriate when the structure is subjected to a transient force as a pulse. This is commonly the case, for example, in seismology. The mathematical model in such cases is an initial-boundary-value problem for a coupled system of wave equations. This hyperbolic problem is different mathematically from the elliptic problem associated with the time-harmonic model and hence requires different computational methods for its solution. We shall very briefly discuss some of these methods as well as some ways of potentially speeding up the computations.

### *Boundary Conditions*

As for the time-harmonic model, there are different formulations of the radiation condition on an artificial outer boundary. It is important to choose the most appropriate one for a given problem. Typical formulations for the time-dependent model are based on local, approximate absorbing boundary conditions. Viscous damper boundary conditions such as those described in Eq. (5), are commonly

used. However, they require that the artificial boundary be placed sufficiently far away from the source of excitation so that reflections are not pernicious. A hierarchy of local, approximate boundary conditions was developed in Ref. 13 for the acoustic-wave equation and in Ref. 29 for the elastic-wave equation. As in the case of the viscous boundary condition, the accuracy of these boundary conditions depends on the deviation of the wave from normal incidence at the artificial boundary. The higher order conditions are more accurate, and hence the size of the computational domain may be decreased. However, they are more difficult to implement.

Finally, we mention a boundary-condition approach developed in Ref. 30 to eliminate reflections for transient problems. This method is based on the calculation of two independent solutions in which the reflections are of opposite sign. The addition of these solutions cancels the reflections, leaving only the energy originally incident on the boundary. The two solutions may be obtained, for example, by applying homogeneous Dirichlet and Neumann boundary conditions.

### *Discretization Methods*

We next consider the question of discretizing the resulting initial-boundary-value problem. Now we must consider the time discretization in addition to the spatial discretization. As before the spatial variables may be discretized using either a finite difference or finite element method. As for the time discretization, empirical experience indicates that explicit finite difference time discretizations are more efficient than implicit ones in spite of stability constraints (see Refs. 8 and 31), since the latter method involves the solution of linear equations at each time step. Explicit formulations express the value of a variable at some point at a future time in terms of the value of the variable at that point and neighboring points at the present time and past times. Hence, it is no longer necessary to solve large systems of linear equations. The large computation time for these problems is due mainly to the large number of time steps required and the large number of spatial variables.

We propose two methods for reducing these computational costs. The first suggestion is the use of an adaptive discretization method. In a previous section we described the potential advantages associated with such methods. Adaptive methods appropriate for hyperbolic problems are different from those most useful for elliptic problems. See Ref. 32 and the references cited there for a discussion of adaptive discretization methods for time-dependent problems. The second suggestion is the use of higher order discretization methods in the spatial variables. As demonstrated in Ref. 33, significant gains can be expected from the use of higher order methods. These include higher order finite difference and finite element methods as well as spectral methods. Recent work [34] dealing with the discretization error as the number of wavelengths increases (i.e., at high frequencies or in large domains) indicates that, particularly in these cases, higher order spatial discretization can yield substantial improvements for both transient and stationary propagation models. On the other hand, as indicated in Ref. 33, it often suffices to use a second-order time-discretization method.

## CONCLUSIONS

We have considered the numerical solution of direct deterministic underwater acoustic-propagation problems using finite difference and finite element methods. We first discussed results obtained after implementing a finite element code on a VAX 11/780 and modifying it to treat a propagation model based on the two-dimensional Helmholtz equation with a variable sound speed. An important feature of the code is the implementation of a recently developed iterative method based on the preconditioned conjugate gradient method for solving the resulting large, sparse, indefinite, non-self-adjoint system of linear equations. Since large matrices do not have to be stored or inverted, we were able to solve efficiently over 35,000 complex equations. Furthermore, simple changes in the code were outlined that would substantially increase both storage capabilities and speed.

We next discussed various issues involved in the numerical solution of general two- and three-dimensional propagation models. Such models involve the coupling of the acoustic-wave equation in a fluid with the elastic-wave equation in a solid and the specification of appropriate interface and boundary conditions. This allows for the computational modeling of general ocean environments, including complicated bottom structures as well as a layer of ice on the surface. We considered time-harmonic (stationary) formulations, which commonly occur in underwater acoustic models, as well as time-dependent formulations corresponding to a transient force such as a pulse. The resulting models may be solved approximately by finite difference or finite element methods.

The numerical methods for treating stationary and transient problems are very different. For transient problems, explicit finite difference time discretizations appear to be more efficient than implicit ones, since the latter involve the solution of linear equations at each time step. For stationary problems, we recommend the use of iterative methods over direct methods. For both stationary and transient formulations we recommend the use of finite element methods for the spatial discretization because of their great flexibility in modeling complicated problems.

The size of the problem depends to a large extent on the frequency and the size of the computational domain. In particular, because the waves must be resolved in all directions, the number of degrees of freedom in two (or three) dimensions generally increases as at least the square (or cube) of the frequency. The size of the problem that can be efficiently treated depends on the computer power available as well as the mathematical modeling techniques and the numerical algorithm employed. Vector computers are expected to improve current computer power by approximately a factor of ten over the next few years. Longer term research in multiprocessor systems is intended to result in improvements of two and three orders of magnitude. The numerical methods we have discussed based on finite elements or finite differences are well suited to parallel computation.

Furthermore, we described several mathematical modeling and numerical investigations that are expected to result in substantial improvements in computational efficiency. These include the use of methods for modeling the radiation boundary condition so as to reduce the size of the truncated computational domain, the use of techniques for improving the iterative method, and the use of adaptive discretization methods. Numerical methods can also be useful for assessing the accuracy of various simplified propagation models. We feel that, at the present time, a variety of complicated, realistic, two-dimensional propagation problems can be efficiently solved by numerical methods without the necessity of resorting to simplified models. Furthermore, in view of anticipated improvements in computer power and numerical methods, it is likely that the same will be true for three-dimensional and much larger two-dimensional problems within a few years.

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